- We have X. Let $Y = a + bX$. $(b \neq 0)$

If $E_1 = \{a + bX \leq y\}$ and $E_2 = \{X \leq \frac{y-a}{b}\}$, $E_1 = E_2$ (Same event)

$$F_Y(y) = P(Y \leq y) = P(a + bX \leq y) = P\left(X \leq \frac{y-a}{b}\right) \quad (\text{Here we assume: } b > 0)$$
$$= F_X\left(\frac{y-a}{b}\right)$$

Otherwise, $P(a+bX \leq y) = P\left(X \geq \frac{y-a}{b}\right) = 1 - F_X\left(\frac{y-a}{b}\right)$

Now, $P(X \leq \eta_X(p)) = p \Leftrightarrow F_X(\eta_X(p)) = p$

and $P(Y \leq \eta_Y(p)) = p \Leftrightarrow F_Y(\eta_Y(p)) = p \Leftrightarrow F_X\left(\frac{\eta_Y(p) - a}{b}\right) = p$

$$\Rightarrow \eta_X(p) = \frac{\eta_Y(p) - a}{b}$$

$$\Rightarrow \eta_Y(p) = a + b\eta_X(p)$$

In particular, $\tilde{\mu}_Y = a + b.\tilde{\mu}_X$ (Put $p = 0.5$)

We know: If $X \sim \mathcal{U}(0,1)$, then $Y = a + (b-a).X \sim \mathcal{U}(a,b)$

$$\Rightarrow \eta_Y(p) = a + (b-a).\eta_X(p)$$

$$F_X(x) = \begin{cases} 0, & \text{if } x < 0 \\ x, & \text{if } 0 \leq x \leq 1 \\ 1, & \text{if } x > 1 \end{cases} \Rightarrow F_X(\eta_X(p)) = p \quad [0 < p < 1]$$

gives us $\eta_X(p) = p$

(Solve for x: $F_X(x) = p$)

$$\boxed{\eta_Y(p) = a + (b-a).p}$$

In particular, $\tilde{\mu}_Y = a + (b-a)(.5) = \frac{a+b}{2}$. (as $\tilde{\mu}_X = 0.5$)

- **Normal Distⁿ:** (Gaussian distⁿ)

$X \sim N(\mu, \sigma^2)$, if

$\underbrace{}_{\text{parameters}}$ parameters $(-\infty < \mu < \infty, \sigma^2 > 0) \rightarrow$ 2-parameter family

$$f(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}.\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \forall x \in \mathbb{R}$$

Clearly, $f(x) > 0 \quad \forall x \in \mathbb{R}$

Fact: $\int_{-\infty}^{\infty} f(x; \mu, \sigma^2)\, dx = 1$

and, $\boxed{E(X) = \mu \text{ and } V(X) = \sigma^2.}$

CDF: $F(x; \mu, \sigma^2) = \int_{-\infty}^{x} f(y; \mu, \sigma^2)\, dy$

Properties:

① Symmetry: $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$$f(\mu + x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} = f(\mu - x), \forall x \in \mathbb{R}$$

$\therefore f$ is symmetric about μ.

So, $E(X) = \tilde{\mu} = \mu$.

② Shape of the Curve:

Defⁿ: If $\mu = 0, \sigma^2 = 1 \rightarrow$ Standard Normal distⁿ (Typically Z is used for a standard Normal RV)

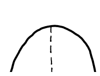
Notation: $f(x; 0, 1) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}} \overset{\text{Notⁿ}}{=} \phi(x)$

$F(x; 0, 1) = \int_{-\infty}^{x} \phi(z)\, dz \overset{\text{Notⁿ}}{=} \Phi(x)$

$f(x; \mu, \sigma^2) = \frac{1}{\sigma} \phi\left(\frac{x-\mu}{\sigma}\right)$

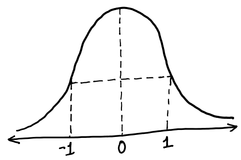
$F(x; \mu, \sigma^2) = \int_{-\infty}^{x} \frac{1}{\sigma} \phi\left(\frac{z-\mu}{\sigma}\right) dz = \int_{-\infty}^{\frac{x-\mu}{\sigma}} \frac{1}{\sigma} \phi(y) \cdot \sigma dy \cdot \left(\text{Put } y = \frac{z-\mu}{\sigma}\right)$

$= \int_{-\infty}^{\frac{x-\mu}{\sigma}} \phi(y)\, dy = \Phi\left(\frac{x-\mu}{\sigma}\right).$



$\phi'(0) = 0$: Point of maxima

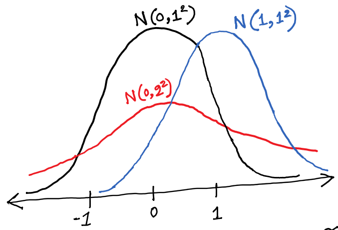
$\phi''(1) = \phi''(-1) = 0$: Points of inflection



$\phi'(0) = 0$: Point of maxima

$\phi''(1) = \phi''(-1) = 0$: Points of inflection

In general, $f'(\mu) = 0$, $f''(\mu+\sigma) = f''(\mu-\sigma) = 0$



Comparison of different normal curves according to their μ and σ^2 values:
Changing μ amounts to shifting the entire curve along the horizontal axis without changing its shape. Changing σ^2 doesn't move the curve along the horizontal axis. Increasing σ^2 makes the curve shorter and wider, while decreasing σ^2 makes it taller and narrower.

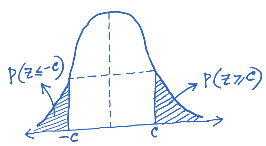
③ CDF:

$$1 - \Phi(c) = P(Z \geq c) = \int_c^{\infty} \phi(z)\,dz \quad \left[\text{Recall: } \overbrace{\phi(z) = \phi(-z)}^{\text{Even fn.}} \text{ by symmetry at } 0\right]$$

$$= \int_{-c}^{-\infty} \phi(-u)(-du) \quad [\text{Put } u = -z]$$

$$= \int_{-\infty}^{-c} \phi(u)\,du$$

$$= P(Z \leq -c) = \Phi(-c) \quad \forall c \in \mathbb{R}$$



$$\boxed{\Phi(-c) = 1 - \Phi(c)}$$

In general, if X is symmetric about μ,

$$P(X \geq \mu + c) = P(X \leq \mu - c)$$

Eg.

$$P(Z \leq 1.25) = \Phi(1.25) = 0.8944$$

$$P(Z \geq 1.25) = 1 - \Phi(1.25) = 0.1056$$

$$P(Z \leq -1.25) = \Phi(-1.25) = 1 - \Phi(1.25) = 0.1056$$

$$P(-0.38 \leq Z \leq 1.25) = \Phi(1.25) - \Phi(-0.38)$$

$$= 0.8944 - 0.3520$$

$$= 0.5424$$

$$P(-1.25 \leq Z \leq 0.38) = \Phi(0.38) - \Phi(-1.25) = (1 - \Phi(-0.38)) - (1 - \Phi(1.25))$$

$$= \Phi(1.25) - \Phi(-0.38)$$

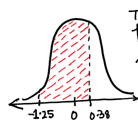
$$= 0.6480 - 0.1056$$

$$= 0.5424$$

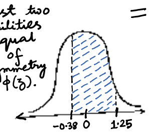
Facts:

1. $P(a \leq X \leq b) = P(a < X \leq b)$
$= P(a \leq X < b) = P(a < X < b)$
$= P(X \leq b) - P(X \leq a)$
$= F_X(b) - F_X(a)$.

2. $P(X \leq a) = P(X < a) = 1 - P(X \geq a)$
$= 1 - P(X > a)$
$= F_X(a)$.



The last two probabilities are equal because of the symmetry of $\phi(z)$.



Lecture 22

Friday, July 14, 2017 1:00 PM

④ **Moments:** The m^{th} moment of X : $E(X^m)$.

1^{st} moment : $\mu = E(X)$.

For Standard Normal Z, $\mu = 0$. (by Symmetry)

i.e. $\phi(z) = \phi(-z)$: Even function

$$E(Z) = \int_{-\infty}^{\infty} z\phi(z)\,dz = \int_{-\infty}^{0} z\phi(z)\,dz + \underbrace{\int_{0}^{\infty} z\phi(z)\,dz}_{} = 0$$

$$= \int_{0}^{-\infty} (-u)\phi(-u)(-du) \quad [\text{Put } u = -z]$$

$$= \int_{0}^{-\infty} u\phi(u)\,du$$

$$= -\int_{-\infty}^{0} z\phi(z)\,dz$$

Here, $h(z) = z\phi(z)$ is odd

as $h(-z) = -z\phi(-z)$
$= -z\phi(z)$
$= -h(z)$

Similarly, $g(z) = z^m\phi(z)$ is odd when m is odd. [m is odd]

As $g(-z) = (-z)^m\phi(-z) = -z^m\phi(z) = -g(z)$

So, $E(Z^m) = 0$, $\forall\, m = 1, 3, 5, \ldots$

We know: $V(Z) = 1 \Rightarrow E(Z^2) - E^2(Z) = 1 \Rightarrow E(Z^2) = 1$ (In general, $E(Z^m)$ is not easy to find for even m.)

⑤ **linear function of Normal RV:**

If $X \sim N(\mu, \sigma^2)$, then $aX + b \sim N(a\mu + b, a^2\sigma^2)$ (Take $a > 0$)

Similar proof can be made for $a < 0$.

Let $Y = aX + b$.

$$F_Y(y) = P(Y \le y) = P(aX + b \le y) = P\left(X \le \frac{y-b}{a}\right)$$

$$= \int_{-\infty}^{\frac{y-b}{a}} \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx$$

$$= \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{\left(\frac{z-b}{a} - \mu\right)^2}{2\sigma^2}} \cdot \frac{dz}{a} \quad [\text{Put } z = ax + b]$$

So, $a\left(\frac{y-b}{a}\right) + b = y$ (upper limit)
and, $a(-\infty) + b = -\infty$ (lower limit) } as $a > 0$
And $\frac{dz}{dx} = a$.

$$= \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi(a^2\sigma^2)}} \cdot e^{-\frac{(z - a\mu - b)^2}{2a^2\sigma^2}}\,dz$$

$$= F(y;\, a\mu + b,\, a^2\sigma^2)$$

We know:
$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(y-\mu)^2}{2\sigma^2}}\,dy$$
as $X \sim N(\mu, \sigma^2)$.

$$F(x;\mu,\sigma^2) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-\mu)^2}{2\sigma^2}}\,dy$$

$$\Rightarrow Y = aX + b \sim N(a\mu + b, a^2\sigma^2).$$

Special Cases: ① If $X \sim N(\mu, \sigma^2)$, then $Z = \frac{X - \mu}{\sigma} \sim N(0, 1)$ [Standardization]

$\left(a = \frac{1}{\sigma},\ b = -\frac{\mu}{\sigma}\right)$

② $Z \sim N(0,1) \Rightarrow \mu + \sigma Z \sim N(\sigma \cdot 0 + \mu = \mu,\ \sigma^2 \cdot 1 = \sigma^2)$

$(a = \sigma,\ b = \mu)$

③ **CDF (contd.):**

Let $X \sim N(\mu, \sigma^2)$,

Let $F(x) = P(X \le x)$

Now, $P(X \le a) = P(X < a) = F(a) = P\left(\frac{X-\mu}{\sigma} \le \frac{a-\mu}{\sigma}\right)$
$= P\left(Z \le \frac{a-\mu}{\sigma}\right)$
$= \Phi\left(\frac{a-\mu}{\sigma}\right)$

$P(a \le X \le b) = P(a < X \le b) = P(a \le X < b) = P(a < X < b) = F(b) - F(a)$
$= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$.

$P(X > a) = P(X \ge a) = 1 - F(a) = 1 - \Phi\left(\frac{a-\mu}{\sigma}\right)$.

Empirical Rule:

Put $b = \mu + \sigma$ and $a = \mu - \sigma$

$P(\mu - \sigma \le X \le \mu + \sigma) = \Phi(1) - \Phi(-1)$
$= 0.8413 - 0.1587$
$= 0.6826$ ($\approx 68\%$)

Similarly, $P(\mu - 2\sigma \le X \le \mu + 2\sigma) = 0.9544$ ($\approx 95\%$)
$P(\mu - 3\sigma \le X \le \mu + 3\sigma) = 0.9974$ ($\approx 99.7\%$)

⑥ **Percentiles:** z_α : $(100p)^{th}$ upper percentile (in the book)

i.e. $P(Z \ge z_\alpha) = \alpha$: $(100(1-\alpha))^{th}$ percentile
$= \eta(1-\alpha)$

To be consistent, we'll call them $z_{1-\alpha}$.

So $P(Z \le z_\alpha) = \alpha$ (For us) or, $z_\alpha = \eta(\alpha)$.

Let $\alpha = .975$, $P(Z \le z_{.975}) = \Phi(z_{.975}) = .975$

$\Rightarrow z_{.975} = 1.96$ (From Table)

Similarly, $z_{.95} = 1.645$ (From Table)
$z_{.99} = 2.33$

Now, $P(-1.96 \le Z \le 1.96) = \Phi(1.96) - \Phi(-1.96)$
$= 2\Phi(1.96) - 1$ (as $\Phi(-1.96) = 1 - \Phi(1.96)$)

$= 0.975 \times 2 - 1 = 0.95$

Similarly, $P(-1.645 \le Z \le 1.645) = 0.9$

$P(-2.33 \le Z \le 2.33) = 0.98$

Let $X \sim N(\mu, \sigma^2) \Rightarrow X = \mu + \sigma Z$

$\Rightarrow \eta_X(p) = \mu + \sigma . \eta_Z(p)$

$= \mu + \sigma . z_p$

$$\boxed{\eta_X(\alpha) = \mu + \sigma . z_\alpha}$$

Eg. Birth Weights (in grams) $\sim N(3432, 482^2)$

(a) $P(3000 \le X \le 4000) = P\left(\frac{3000-3432}{482} \le \frac{X-3432}{482} \le \frac{4000-3432}{482}\right)$

"Forward Problem"

$= P(-0.90 \le Z \le 1.18)$

$= \Phi(1.18) - \Phi(-0.90)$

$= 0.8810 - 0.1841$ (From Table)

$= 0.6969$

69.69% of babies have birth weights between 3 kgs. and 4 kgs.

(b) What is $\eta(.999)$? ("Backward Problem")

$\eta(.999) = 3432 + 482 . z_{.999}$

$= 3432 + 482 \times 3.10$ (From Table)

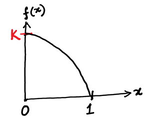
$= 4926.2$

0.1% of babies will have birth weights of 4.9262 kgs. or more.

Lecture 23

Monday, July 17, 2017 12:59 PM

- $f(x) = \begin{cases} K(1-x^2) & , 0 \le x \le 1 \\ 0 & , \text{o.w.} \end{cases} = K(1-x^2)\, I_{[0,1]}(x)$



$D = [0,1]$

$$\int_0^1 f(x)\,dx = \int_0^1 K(1-x^2)\,dx = K\left[x - \frac{x^3}{3}\right]_0^1 = K\left(1 - \frac{1}{3}\right) = \frac{2K}{3} = 1$$

$$\Rightarrow K = \frac{3}{2}$$

$$F(x) = \begin{cases} 0 & , \text{ if } x \le 0 \\ \int_0^x \frac{3}{2}(1-y^2)\,dy = \frac{3}{2}x - \frac{1}{2}\cdot x^3 & , \text{ if } 0 < x < 1 \\ 1 & , \text{ if } x \geqslant 1 \end{cases}$$

$\eta(0.5)$: $F(x) = 0.5 \Rightarrow \frac{3}{2}\cdot x - \frac{1}{2}x^3 = \frac{1}{2} \Rightarrow 3x - x^3 = 1$

Solve for x : $\tilde{\mu} = 0.347$ (Unique root in $[0,1]$ – Plot the fn.)

Sec 4.4/5: • Lognormal dist^n:

Def^n: $Y \sim LN(\mu, \sigma^2)$ if $\log_e Y \sim N(\mu, \sigma^2)$ $\left[\begin{matrix} -\infty < \mu < \infty \\ \sigma^2 > 0 \end{matrix}\right]$

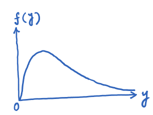
$D_Y = (0, \infty)$

CDF of Y: $F_Y(y) = P(Y \le y) = P(\log_e Y \le \log_e y) = \int_{-\infty}^{\log_e y} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx$ [As $\log Y \sim N(\mu, \sigma^2)$]

[Take $y > 0$; Clearly, $P(Y \le y) = 0$ for $y \le 0$]

$$= \int_0^y \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\log_e \omega - \mu)^2}{2\sigma^2}} \cdot \frac{d\omega}{\omega}$$ [Put $x = \log_e \omega$, so $dx = \frac{d\omega}{\omega}$]

$$\Rightarrow f(y; \mu, \sigma^2) = \begin{cases} \frac{1}{\sqrt{2\pi\sigma^2}\cdot y} e^{-\frac{(\log y - \mu)^2}{2\sigma^2}} & ; \text{ if } y > 0 \\ 0 & ; \text{ o.w.} \end{cases}$$



$$P(Y \le \eta(p)) = p$$

$$\Rightarrow P(\log_e Y \le \log_e \eta(p)) = p$$

$$\Rightarrow P\left(\frac{\log_e Y - \mu}{\sigma} \le \frac{\log_e \eta(p) - \mu}{\sigma}\right) = p$$

$$\Rightarrow P\left(Z \le \frac{\log_e \eta(p) - \mu}{\sigma}\right) = p$$ [As $\log_e Y \sim N(\mu, \sigma^2)$]

$$\Rightarrow \frac{\log_e \eta(p) - \mu}{\sigma} = z_p$$

$$\Rightarrow \boxed{\eta(p) = \exp\{\mu + \sigma z_p\}}$$

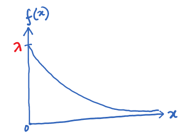
• Exponential Dist^n:

- **Exponential Distⁿ:**

Defⁿ: $X \sim Exp(\lambda)$; $\lambda > 0$ if

$$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x} & ; \text{ if } x > 0 \\ 0 & ; \text{ o.w.} \end{cases}$$

Used to model waiting times of events occurring randomly over time.



$$F(x) = \begin{cases} 0 & , \text{ if } x \le 0 \\ 1 - e^{-\lambda x} & , \text{ if } x > 0. \end{cases}$$

$E(X) = \frac{1}{\lambda}$, $V(X) = \frac{1}{\lambda^2}$. (Check)

$P(X \geq x) = e^{-\lambda x}$, if $x > 0$

$$P(X \geq t + t_0 \mid X \geq t_0) = \frac{P(X \geq t + t_0)}{P(X \geq t_0)} \quad [\text{Where } t, t_0 > 0]$$

$$= \frac{e^{-\lambda(t+t_0)}}{e^{-\lambda t_0}} = e^{-\lambda t} = P(X \geq t)$$

: Memoryless Property — Distⁿ of remaining waiting time is NOT affected by how long you have waited.

$P(X \geq t + t_0 \mid X \geq t_0) \neq P(X \geq t + t_0)$: Memoryless is NOT indep. of the events $\{X \geq t + t_0\}$ and $\{X \geq t_0\}$.

Verify that this happens for Geo(p) also. In fact, these are the only two distⁿˢ with this property.

- **Chapter 5 :** (Joint Prob. Distⁿ:)

E.g.: Among the customers who buy both homeowner and auto policies from an insurance company,

Let us define:

X = deductible on the auto policy

Y = deductible on the homeowner policy

Table of $p(x,y) : P(X = x, Y = y)$

x \ y	0	100	200
100	0·2	0·1	0·2
250	0·05	0·15	0·3

$p(100, 100) = P(X = 100, Y = 100) = P(\$100 \text{ deductible on both policies})$

$= 0 \cdot 1$

For two discrete r.v.'s X and Y, joint distⁿ describes prob. for each pair (x, y) [where, $x \in D_X$, $y \in D_Y$]

Joint p.m.f.: $p(x,y) \overset{\text{defn.}}{=} P(X = x, Y = y)$

Lecture 24

Tuesday, July 18, 2017 1:00 PM

Joint pmf (of two discrete RV's X & Y) $:\overset{defn}{=} P(X = x, Y = y)$

$\overset{Not^n}{=} p(x,y)$ [$= 0$ if $x \notin D_X$ or $y \notin D_Y$]

$p(x,y)$	y: 0	100	200	$p_X(x)$
x: 100	0.2	0.1	0.2	0.5
250	0.05	0.15	0.3	0.5
$p_Y(y)$	0.25	0.25	0.5	1

$D = \{(100,0), (100,100), (100,200), (250,0), (250,100), (250,200)\}$

Joint Support of (X,Y) $:\overset{defn}{=} D = \{(x,y) : p(x,y) > 0\}$

• Requirement: $\sum_{x \in D_X} \sum_{y \in D_Y} p(x,y) = 1 \iff \sum\sum_{(x,y)\in D} p(x,y) = 1$

and $p(x,y) \geq 0 \;\; \forall (x,y)$

$P((X,Y) \in A) = \sum\sum_{(x,y)\in A} p(x,y)$ [A is a set of pairs (x,y)]

Eg: P(at least \$100 deductible on both policies)

$= P(X \geq 100, Y \geq 100) = P(\min(X,Y) \geq 100)$

$= p(100,100) + p(100,200) + p(250,100) + p(250,200)$

$= 0.75$

[Take $A = \{(x,y) : x \geq 100, y \geq 100\} = \{(x,y) : \min(x,y) \geq 100\}$]

$P(X = 100) = p(100,0) + p(100,100) + p(100,200) = 0.5$

$P(X = 250) = 1 - 0.5 = 0.5$

$\hookrightarrow P(X = 250, -\infty < Y < \infty)$

Defn: Marginal p.m.f of X & Y:

$p_X(x) = \sum_{y \in D_Y} p(x,y)$, $p_Y(y) = \sum_{x \in D_X} p(x,y)$

$= \ldots p(250,200) = 0.75$

$$p_X(x) = \sum_{y \in D_Y} p(x,y) \,, \quad p_Y(y) = \sum_{x \in D_X} \ldots$$

Eg: $$P(Y \geqslant 100) = \underline{p(100,100)} + \underline{p(100,200)} + \underline{p(250,100)} + \underline{p(250,200)} = 0.75$$

$$= p_Y(100) + p_Y(200) = P(Y=100) + P(Y=200)$$

Same as $P(\underbrace{X \geqslant 100}_{\text{Certain Event}}, Y \geqslant 100)$

Here, $p_Y(0) = p_Y(100) = 0.25$, $p_Y(200) = 0.5$

Similarly, $p_X(x) = 0.5$ for $x = 100, 250$.

Find out the mean and variance of X, when X has an exponential distribution with parameter λ (> 0).

$$E(X) = \int_0^\infty \lambda x e^{-\lambda x}\,dx = \int_0^\infty y e^{-y}\,\frac{dy}{\lambda} \quad (\text{Putting } y = \lambda x)$$

$$= \frac{1}{\lambda}\int_0^\infty y e^{-y}\,dy$$

$$= \frac{1}{\lambda}\cdot\left[-y e^{-y}\right]_0^\infty + \frac{1}{\lambda}\int_0^\infty e^{-y}\,dy \quad (\text{Integrating by parts})$$

$$= \frac{1}{\lambda}\cdot 0 + \frac{1}{\lambda}\left[-e^{-y}\right]_0^\infty = \frac{1}{\lambda}$$

$$E(X^2) = \int_0^\infty \lambda x^2 e^{-\lambda x}\,dx = \int_0^\infty \lambda\cdot\frac{y^2}{\lambda^2} e^{-y}\,\frac{dy}{\lambda} \quad (\text{Putting } y = \lambda x)$$

$$= \frac{1}{\lambda^2}\int_0^\infty y^2 e^{-y}\,dy = \frac{1}{\lambda^2}\left[-y^2 e^{-y}\right]_0^\infty + \frac{2}{\lambda^2}\int_0^\infty y\cdot e^{-y}\,dy \quad (\text{Integrating by parts})$$

$$= \frac{1}{\lambda^2}\cdot 0 + \frac{2}{\lambda^2}\cdot 1 \quad (\text{From earlier part})$$

$$= \frac{2}{\lambda^2}$$

$$\therefore V(X) = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

Prove that the n^{th} moment of the log-normal distribution is $e^{n\mu+\frac{n^2}{2}\sigma^2}$. Then use that to show that the mean and variance of the log-normal distribution are $e^{\mu+\frac{1}{2}\sigma^2}$ and $e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)$ respectively.

Hints: 1. If $Y \sim LN(\mu, \sigma^2)$, then write Y as a function of some $Z \sim N(0,1)$, say $g(Z)$.

Then $E(Y^n) = E[\{g(Z)\}^n]$ and you can evaluate the expectation using the standard normal density $\varphi(z)$ instead of the density $f(y)$ of the log-normal distribution.

2. The integral $\int_{-\infty}^{\infty} e^{at^2+bt}dt$ can be simplified by completing the square as:

$$\int_{-\infty}^{\infty} e^{at^2+bt}dt = \int_{-\infty}^{\infty} e^{a(t^2+\frac{b}{a}t)}dt = e^{\frac{-b^2}{4a}}\int_{-\infty}^{\infty} e^{a(t^2+2\frac{b}{2a}t+\frac{b^2}{4a^2})}dt = e^{\frac{-b^2}{4a}}\int_{-\infty}^{\infty} e^{a(t+\frac{b}{2a})^2}dt.$$

Now if $a < 0$, you can identify the integrand as $e^{\frac{-(t-\mu)^2}{2\sigma^2}}$ for some μ and $\sigma > 0$ and be able to find the value of the integral.

Let $Y \sim LN(\mu, \sigma^2)$

$\Rightarrow \log_e Y \sim LN(\mu, \sigma^2)$

$\Rightarrow Z = \dfrac{\log_e Y - \mu}{\sigma} \sim N(0,1)$

where, $Y = e^{\mu + \sigma Z}$

$$\text{Now, } E(Y^n) = E\left(e^{n\mu + n\sigma Z}\right) = \int_{-\infty}^{\infty} e^{n\mu + n\sigma Z} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{Z^2}{2}} \, dZ$$

$$= \int_{-\infty}^{\infty} \frac{e^{n\mu}}{\sqrt{2\pi}} \cdot e^{-\frac{Z^2}{2} + n\sigma Z} \cdot dZ$$

$$= \frac{e^{n\mu}}{\sqrt{2\pi}} \cdot e^{-\frac{n^2\sigma^2}{4\times(-\frac{1}{2})}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(Z + \frac{n\sigma}{2\times(-\frac{1}{2})}\right)^2} dZ$$

(using hint 2 and completing the square with $a = -\frac{1}{2}$, $b = n\sigma$)

$$= \frac{e^{n\mu + \frac{n^2\sigma^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\cdot(Z - n\sigma)^2} dZ$$

$$= e^{n\mu + \frac{n^2\sigma^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(Z-n\sigma)^2}{2}} dZ$$

$$= e^{n\mu + \frac{n^2\sigma^2}{2}} \cdot \int_{-\infty}^{\infty} f(Z)\, dZ$$ [where $f(Z)$ is the density of $N(n\sigma, 1)$]

$$= e^{n\mu + \frac{n^2\sigma^2}{2}} (1)$$

$$= e^{n\mu + \frac{n^2\sigma^2}{2}}$$ (Proved)

$$\text{Now, } E(Y) = e^{1\cdot\mu + \frac{1^2\sigma^2}{2}} = e^{\mu + \frac{\sigma^2}{2}}$$

$$V(Y) = E(Y^2) - E^2(Y)$$

$$= e^{2\cdot\mu + \frac{2^2\sigma^2}{2}} - \left(e^{\mu + \frac{\sigma^2}{2}}\right)^2$$

$$= e^{2\mu + 2\sigma^2} - e^{2\mu + \sigma^2}$$

$$= e^{2\mu + \sigma^2} \cdot \left(e^{\sigma^2} - 1\right)$$ (Proved)